Networked Control Systems (ME-427) - Exercise session 3

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1. Consider the following switched system

$$x_{k+1} = A_{\sigma(k)}x_k, \qquad \sigma(k) \in \{1,2\}$$

$$A_1 = \begin{bmatrix} 0.4 & -0.9 \\ 0.3 & 0.5 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.7 & 0.1 \\ 0.3 & 0.6 \end{bmatrix}$$

Is there a common Lyapunov function certifying exponential stability?

Hint: adapt the MatLab code provided in the previous exercise session.

Solution: The MatLab code can be seen on moodle. Here, we explain why we can guarantee exponential convergence with the common Lyapunov function. Given the Lyapunov function P, there exists $Q < 0$ such that, for any $i \in \{1,2\}$, $A_i^\top PA_i - P < Q$. Let $V(k+1) = x^\top(k+1)Px(k+1)$, then $V(k+1) - V(k) < x(k)Qx(k)$. Since $Q < 0$, there exists a scaler $\mu > 0$ such that $Q < -\mu P$. Therefore, $V(k+1) < V(k) - \mu V(k) = (1-\mu)V(k)$ and thus $V(k)$ decreases exponentially. Further considering that there exists a scaler $\rho > 0$ such that $||x(k)||^2 < \rho V(k)$, the magnitude $||x(k)||$ also decreases exponentially.

2. For the switched system

$$x_{k+1} = A_{\sigma(k)}x_k, \qquad \sigma(k) \in \mathcal{I} = \{0,1,\ldots,M\} \tag{1}$$

one might think that if all matrices A_i, $i \in \mathcal{I}$ are Schur, then the zero solution is AS, independently of the switch signal $\sigma(k)$. This is unfortunately false, as shown by this system

$$\mathcal{I} = \{1,2\}, \qquad A_1 = \begin{bmatrix} 0.9901 & 0.1988 \\ -0.0994 & 0.9881 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.9424 & 0.0946 \\ -0.1892 & 0.9405 \end{bmatrix}$$

(a) Check that A_1 and A_2 are Schur.

(b) Consider

$$\sigma_k = \begin{cases} 1 & \text{if } x_k \geq 0 \text{ or } x_k \leq 0 \\ 2 & \text{otherwise} \end{cases}$$

where $x_k \geq 0$ means ($x_{k,1} \geq 0$ and $x_{k,2} \geq 0$). Write the MatLab code for simulating the system from $x(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and plot $x(k)$ in the $x_1 - x_2$ plane.
Analyze the qualitative behavior of $x(k)$ on each orthant for understanding why stability fails.

Solution: see the MatLab code on moodle.

3. Prove that

(a) If $A \in \mathbb{R}^n$ is diagonalizable, (i.e. $A = V^{-1}DV$ where D is a diagonal matrix and V collects the eigenvectors of A as columns), then, for $t \in \mathbb{R}$, it holds $e^{At} = V^{-1}e^{Dt}V$, i.e., A and e^{At} can be diagonalized using the same matrix V.

Hint: use the definition of e^{At}.

Solution: For A^k, $k = 1, 2, \ldots$ one has

$$A^k = \underbrace{(V^{-1}DV)(V^{-1}DV)\cdots(V^{-1}DV)}_{k \text{ times}} = V^{-1}D^kV$$

Using the definition of e^{At}, one has

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2} + \cdots + \frac{(At)^k}{k!} + \cdots = \\ &= V^{-1}V + V^{-1}(tD)V + V^{-1}\frac{(tD)^2}{2}V + \cdots + V^{-1}\frac{(tD)^k}{k!}V + \cdots = \\ &= V^{-1}\left(1 + tD + \frac{(tD)^2}{2} + \cdots + \frac{(tD)^k}{k!} + \cdots\right)V = V^{-1}e^{Dt}V \end{aligned}$$

(b) For

$$A = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} \quad \lambda \in \mathbb{R},$$

one has

$$e^{At} = \frac{1}{2}\begin{bmatrix} e^{t(1+\lambda)} + e^{t(\lambda-1)} & e^{t(1+\lambda)} - e^{t(\lambda-1)} \\ e^{t(1+\lambda)} - e^{t(\lambda-1)} & e^{t(1+\lambda)} + e^{t(\lambda-1)} \end{bmatrix}.$$

Hint: A is symmetric and hence diagonalizable.

Solution: We first compute the eigenvectors of A for eigenvalues $\xi_1 = \lambda - 1$ and $\xi_2 = \lambda + 1$. For ξ_1, one has

$$\begin{gathered} Av_1 = \xi_1 v_1 \\ \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = (\lambda - 1)\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \end{gathered}$$

$$\begin{cases} \lambda v_{11} + v_{12} = (\lambda - 1)v_{11} \\ v_{11} + \lambda v_{12} = (\lambda - 1)v_{12} \end{cases}$$

$$\begin{cases} v_{12} = -v_{11} \\ v_{11} = -v_{12} \end{cases}$$

Hence $v_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$. By similar computations, one has that $Av_2 = \xi_2 v_2$ is verified by $v_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Setting

$$V = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

one has

$$V^{-1} = \frac{1}{2}V$$

Then,

$$e^{At} = \frac{1}{2}Ve^{\left[\begin{smallmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{smallmatrix}\right]t}V = \frac{1}{2}V\left[\begin{smallmatrix} e^{(\lambda-1)t} & 0 \\ 0 & e^{(\lambda+1)t} \end{smallmatrix}\right]V$$

which provides the desired result.